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Shape factors for conductive heat flow in circular and quadratic cross-sections

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Abstract—The two-dimensional steady heat flow in bodies with different cross-sections and isothermal boundaries is a problem that has been investigated intensively in the literature. For the cross-section of two concentric circles the analytical solution of the shape factor as a function of the ratio of the outer diameter to the inner diameter is well known and can be found in most textbooks on heat transfer. For some simple well defined geometries it is possible to calculate the shape factor by the method of conformal mapping. But for arbitrarily shaped cross-sections numerical methods or correlations fitted to measured or calculated values have to be used. We applied the Method of Finite Elements and the Method of Finite Differences to calculate shape factors for cross-sections bounded by concentric circles and squares. By developing the shape factor for the investigated cross-sections analytical approximations for the shape factor are obtained. The approximations fulfil limiting cases and are close to the exact solutions when being fitted to the numerical values. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

To calculate the two-dimensional heat flow $\dot{Q}$ through a solid shell of an arbitrarily shaped cross-section and a length l perpendicular to the cross-section where the two boundaries A_i and A_o are at different but constant temperatures T_i and T_o (see Fig. 1) the temperature distribution has to be known. The problem is described by Fourier's equation for conductive heat flow:

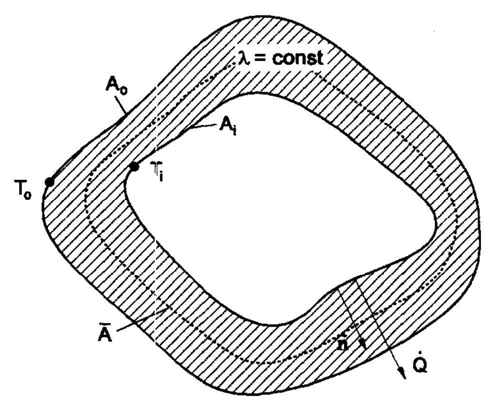

Fig. 1. Arbitrarily shaped cross-section, different but uniform temperatures at the inner and outer surface A_i and A_o.

$$d\dot{Q} = -\lambda \frac{\partial T}{\partial \mathbf{n}} dA. \tag{1}$$

After introduction of a dimensionless temperature

$$\Theta \equiv \frac{T - T_o}{T_i - T_o} \tag{2}$$

equation (1) may be integrated to give

$$\dot{Q} = \lambda \int_A \left(\frac{\partial \Theta}{\partial \mathbf{n}}\right) dA (T_o - T_i). \tag{3}$$

With the shape factor S, introduced by Langmuir [1], defined as

$$S \equiv \int_A \left(\frac{\partial \Theta}{\partial \mathbf{n}}\right) dA, \tag{4}$$

the solution of equation (1) may then be expressed as

$$\dot{Q} = \lambda S (T_o - T_i). \tag{5}$$

For a two-dimensional problem S does not depend on the length l of the object and therefore one may define, by dividing the original definition (4) by l, a shape factor S_l

$$S_l \equiv \frac{l}{l} \int_A \left(\frac{\partial \Theta}{\partial \mathbf{n}}\right) dA, \tag{6}$$

to obtain the heatflux per unit length from equation (5):

$$\dot{Q}/l = \lambda S_l (T_o - T_i). \tag{7}$$

To estimate S_l the shape factor may be interpreted as the ratio of the average surface area $\bar{A}$ for conduction to the average length $\bar{s}$ of the conduction path,

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NOMENCLATURE

A	surface area [m^2]
$\bar{A}$	average surface area [equation (8)] [m^2]
a_i	coefficients in equation (10)
C_i	constants
l	length [m]
n	grid density, see text for details [1]
$\mathbf{n}$	unit vector perpendicular to a surface [m]
$\dot{Q}$	heat flow [W]
R_l	shape resistance ($1/S_l$)
r	radius (Fig. 2) [m]
S_l	shape factor [equation (6)]
S	asymptotic value for the shape factor [equation (10)]
$\bar{s}$	average distance [equation (8)] [m]
T	temperature [K]
x	ratio of the outer diameter r_o to the inner diameter r_i.

Greek symbols

λ	heat conductivity [W/(m K)]
Θ	dimensionless temperature.

Subscripts

anal	analytical
calc	calculated
cc	concentric circles
cs	square within the centre of a circle
i	inner boundary
o	outer boundary
sc	circle within the centre of a square
sp	point within the centre of a square
ss	concentric squares.

$$S_l = \frac{\bar{A}/l}{\bar{s}} = \frac{1}{R_l}, \tag{8}$$

because for the simplest case of a plan plate the heat-flux is proportional to the surface area but reverse proportional to the thickness of the plate.

Instead of S_l its reciprocal R_l, know as the shape resistance, is often used.

Once S_l is known for a two-dimensional shape with constant heat conductivity and each boundary on uniform temperature, the heat flow may be easily calculated by use of equation (7). The problem no longer is the integration of equation (1), but the calculation of S_l from equation (6). This can be done analytically for a few simple shapes [2, 3], but is impossible for most. Empirical correlations are given by Smith *et al.* [4] for the arrangements of two concentric squares or a circle within the centre of a square. These correlations are only valid for a certain range of x defined by

$$x = \frac{2r_o}{2r_i} \tag{9}$$

as the ratio of the outer diameter $2r_o$ to the inner diameter $2r_i$ (see Fig. 2). The parameters of the correlations given by Smith *et al.* are fitted to experiments on electrical flow through paper with constant electrical conductivity. For some values of x numerical results are given. In this work numerical methods are used to calculate the shape factors for three different shaped cross-sections. These are (see Fig. 2):

- concentric squares
- circle within the centre of a square
- square within the centre of a circle.

Analytical approximations for the shape-factor as a function of x are presented. The analytical expressions are obtained by the ratio of the average surface area (logarithmic mean) to an average length of the conduction path (logarithmic, arithmetic or geometric mean). Limiting cases for x tending to infinity and discrete values obtained by numerical calculations with the Method of Finite Elements (FEM) or Finite Differences (DIF) are used to fit constants of the approximations.

2. CALCULATION

The Method of Finite Elements (FEM) is well established to calculate heat transfer problems in a geometry with curved boundaries, whereas the method of Finite Differences (DIF) is often used to calculate heat transfer problems in a geometry with straight boundaries. For FEM the geometry is divided into quadrilateral elements to calculate the temperature distribution and the heat flow through the boundaries. Laplace's equation is solved with the aid of the program package FIDAP Rel. 7.5. In Fig. 3 a mesh with 400 elements for the geometry of concentric squares and $x = 2$ is shown. For reasons of symmetry only one eighth of the shape has to be considered. The calculated heat flow at the inner and outer boundary has to be the same but is different due to errors in numerical integration. A method has to be developed to find the "exact value" of the heat flow and from this value for the shape factor S_l. Calculated values for S_l should be exact and change few for a mesh with the number of elements tending to infinity (high grid density). Therefore an asymptotic function is chosen to describe the dependence of shape-factor on grid density equation (10),

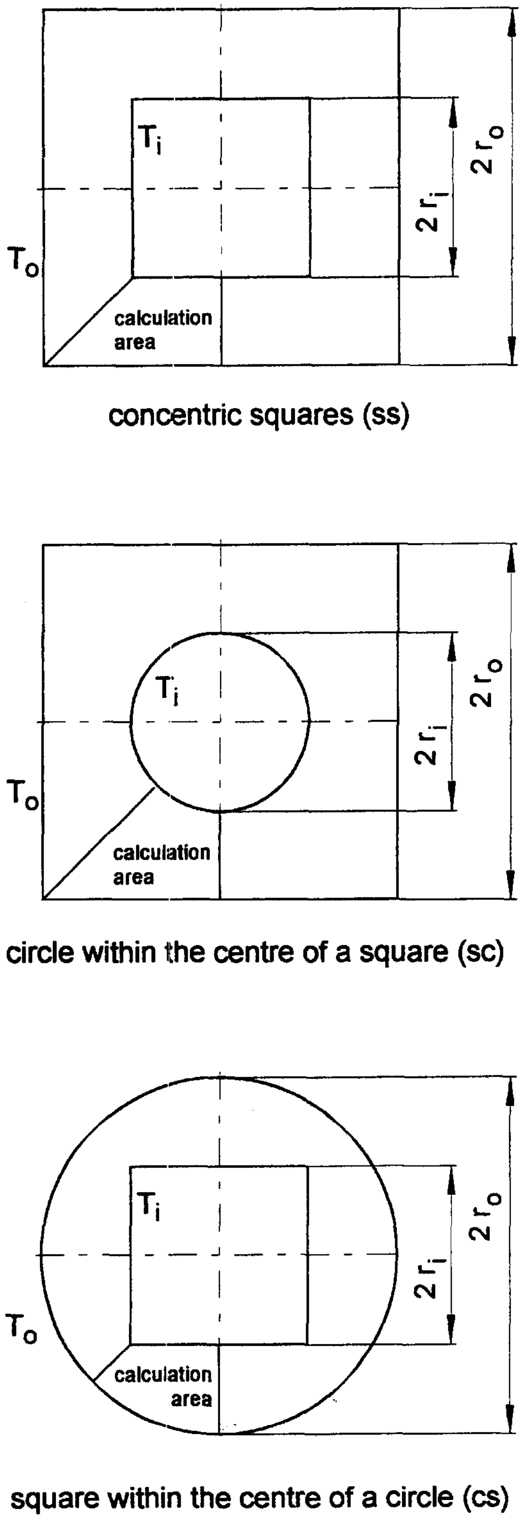

Fig. 2. Considered shapes and area of numerical calculation.

$$S_1 = \widetilde{S}_1 + a_1 \cdot \exp(-a_2 \cdot n^{a_3}) \quad \text{with} \quad \begin{array}{l} a_1 \neq 0 \\ a_2 > 0, \\ a_3 > 0 \end{array} \tag{10}$$

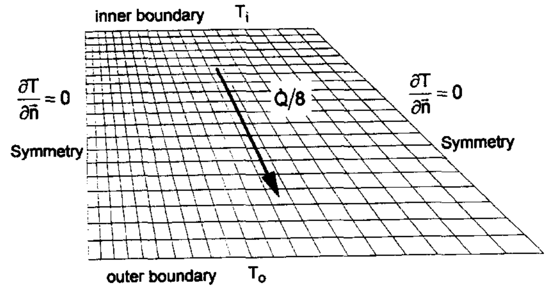

Fig. 3. Mesh plot and boundary conditions for the arrangement of concentric squares with $x = 2$.

where $\widetilde{S}_1$ is the asymptotic limit of the shape factor, assumed to be near the exact value and n is a measure for the grid density. It is defined for Finite Elements as the number of corner-nodes on a boundary edge being equal for all four sides (Fig. 3). The number of elements in the mesh is therefore n^2. For the Finite Differences all elements have the same extension in length. The grid density here is defined as the number of nodes on the inner boundary r_i. For a given ratio x of the outer diameter to the inner diameter different grid densities are used to calculate S_1 from the heat flow at the inner and outer boundary. The coefficients a_i are fitted to the different values of S_1. The values of $\widetilde{S}_1$ so obtained as a function of x are used to fit the analytical approximations.

2.1. *Validation of the numerical methods*

An exact solution for S_1 as a function of x is known for two concentric circles. The average surface area $\bar{A}$ is $2\pi l$ times the logarithmic mean of the outer and inner radii and the length of the conduction path $\bar{s}$ is the difference between inner and outer radius. This leads to

$$S_{1,\mathrm{cc}} = \frac{\bar{A}/l}{\bar{s}} = \frac{2\pi(r_o - r_i)}{\ln\left(\dfrac{r_o}{r_i}\right)(r_o - r_i)} = \frac{2\pi}{\ln(x)}. \tag{11}$$

Discrete values for S_1 as a function of the grid density obtained by numerical calculation are shown in Fig. 4 as symbols. The lines are the fitted functions according to equation (10). The values of $\widetilde{S}_1$ calculated by the described method fulfil the condition that they come close to the exact value obtained from equation (11). Comparing results with the same grid density the numerical values are in better agreement with the exact solution for small values of x than they are for larger ones. The maximum relative absolute error $|S_{1,\mathrm{calc}}/S_{1,\mathrm{anal}} - 1|$ for $1/n \to 0$ was 0.0025 at $x = 10$, with $S_{1,\mathrm{calc}}$ obtained from equation (10). The heat flow at the inner boundary (the circles in Fig. 4) is underestimated whereas the heat flow at the outer boundary (the triangles) is overestimated for low grid densities. Values of S_1 would be calculated too high or too low, respectively, depending on geometry, consideration of inner or outer boundary and numerical method, without

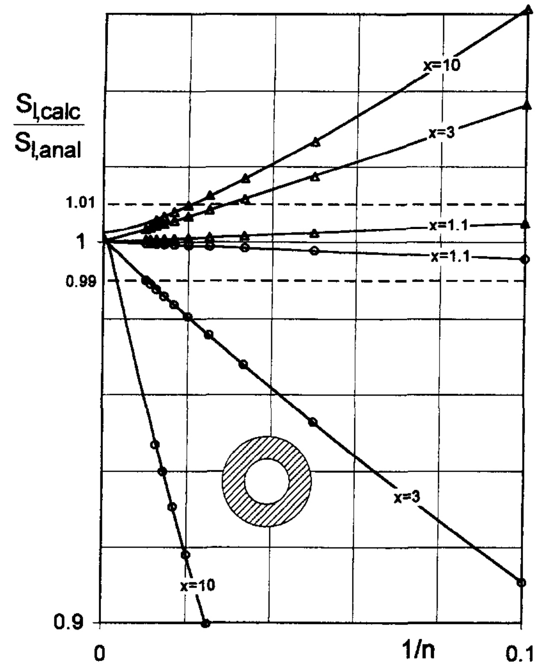

Fig. 4. Shape factors calculated by FEM, divided by the analytical solution from equation (5) as a function of x and the reciprocal of the grid density n for the arrangement of two concentric circles. Triangle: calculated from heat flow at outer boundary. Circle: calculated from heat flow at inner boundary. Lines: equation (4).

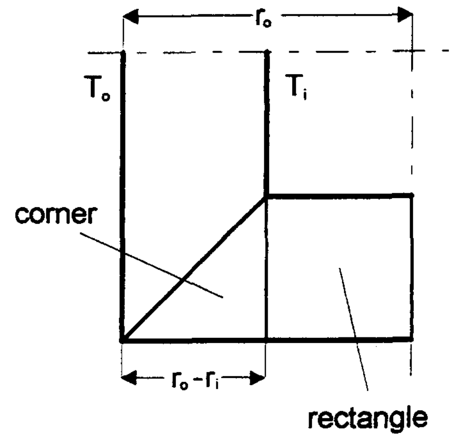

Fig. 5. Geometric consideration for the average length $\bar{s}$.

approximating an asymptotic value. The method described above is used to obtain asymptotic values S_l with the Method of Finite Elements as well as for the Method of Finite Differences. The latter method is only used for the geometry of two concentric squares. In this case the two numerical methods are validated against each other (see Section 3.1).

3. RESULTS

For the three cross-sections shown in Fig. 2 it might be possible to find an analytical solution by conformal-mapping using a Schwarz–Cristoffel integral (the remarks of the reviewer concerning this point, are gratefully acknowledged) but we did not try this method here. The method used is

(1) compute numerical values as described above, which means increase n and fit equation (10), to derive an asymptotic value for S_l,
(2) find a suitable first analytical approximation of S_l from the ratio between the logarithmic mean surface area and a logarithmic, arithmetic or geometric mean length $\bar{s}$ of the conduction path,
(3) take limiting cases into account,
(4) modify some coefficients in the approximation slightly to fit against numerical results with the method of least square minimisation or by minimisation of the maximum relative error.

3.1. *Concentric squares*

The average length $\bar{s}$ is obtained by a simple geometric consideration (see Fig. 5) as a weighted arithmetic mean distance

$$\bar{s} = (r_o - r_i)\Bigg(\underbrace{\frac{r_i}{r_o}\cdot(l)}_{\text{rectangle}} + \underbrace{\left(1-\frac{r_i}{r_o}\right)\cdot\frac{1+\sqrt{2}}{2}}_{\text{corner}}\Bigg) \quad (12)$$

between the shortest way for conduction $(r_o - r_i)$ and the longest $(r_o - r_i)\cdot(1+\sqrt{2})/2$. This gives the following expression for the shape factor.

$$S_{l,ss} \cong \frac{8}{\ln(x)\cdot\left(1+\left(\frac{\sqrt{2}-1}{2}\right)\left(1-\frac{1}{x}\right)\right)}: \quad (13)$$

A comparison with numerical values obtained by the Method of Finite Differences and Finite Elements gives a better fit by modifying (and simplifying) equation (13) to:

$$S_{l,ss} = \frac{8}{\ln(x)\cdot\left(1+\frac{1}{4}\cdot\left(1-\frac{1}{x}\right)\right)}. \quad (14)$$

This equation fits our numerical results with a relative error better than 0.005 for a computed range of x from 1.01 to 10. For FEM calculations the maximum number of Elements in the mesh was 6400 and for DIF calculations n was up to 1500. In the worst case for $x = 10$ the relative difference of the values calculated for S_l by the use of equation (10) to their mean value $(\bar{S}_{l,FEM} + \bar{S}_{l,DIF})/2$ was better than 0.005. In Fig. 6 the shape resistance as a function of x is shown. As a comparison measured and calculated values from Smith *et al.* [4] and Langmuir *et al.* [5] are shown. The two empirical correlations from Smith *et al.* [2–4] being valid only for a certain range of x are fitted to his experimental data and are also shown in Fig. 6. Their numerical values obtained by a relaxation method vary with the number of elements used. The

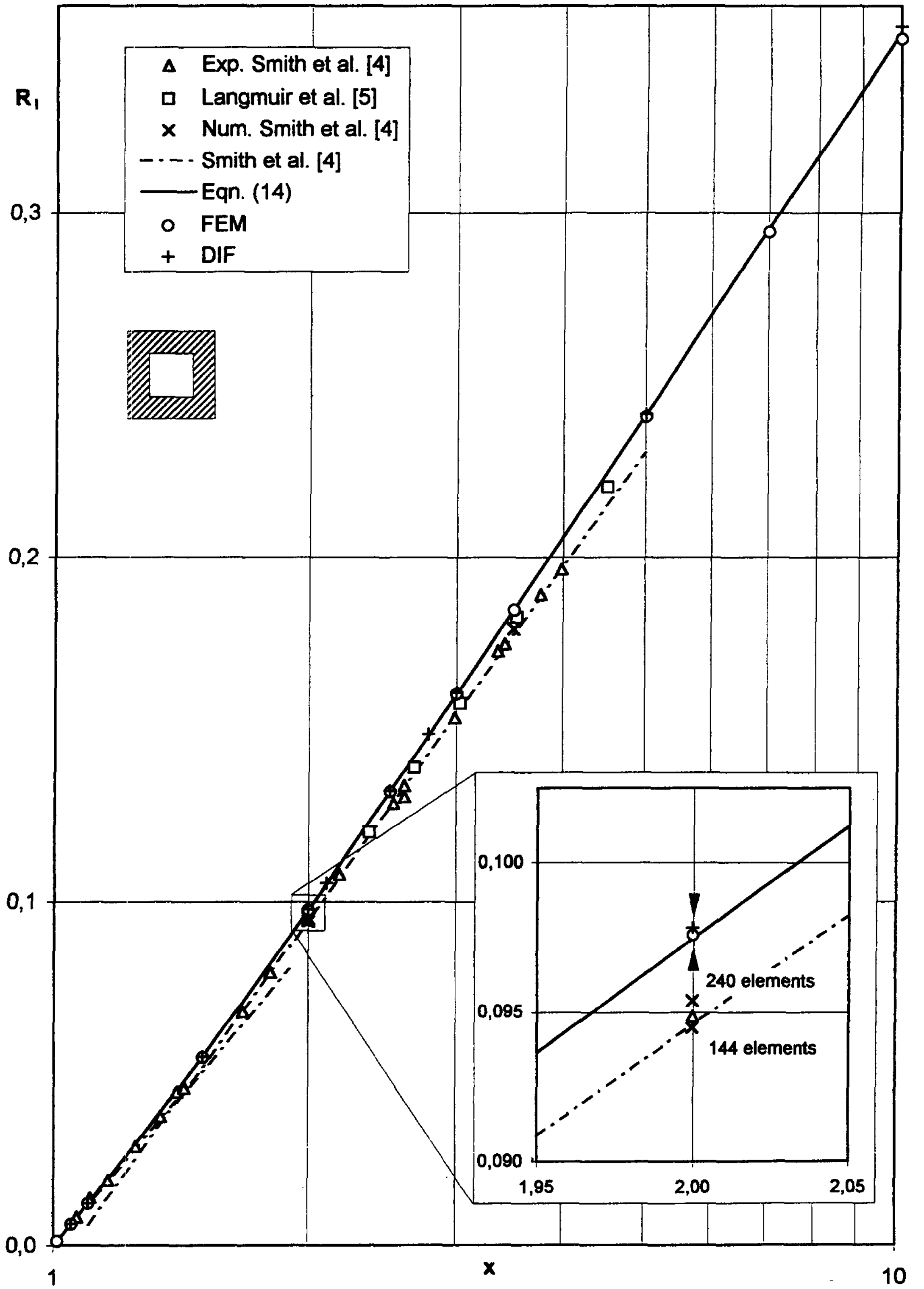

Fig. 6. Shape factor as a function of x for two concentric squares.

difference between their numerical and measured values and our numerical values may be regarded more in detail for $x = 2$. Our numerical values obtained by two different methods do match nearly exactly. The arrows show the tendency of our numerical values with raising grid density toward the asymptotic value of $\widehat{S_1}$ (see Fig. 4). It can be estimated that the numerical value of S_1 of Smith *et al.* should have been lower (higher R_1) with a higher number of elements.

3.2. *Circle within the centre of a square*

The average length $\bar{s}$ in this case cannot be found like for concentric squares, because the circle has no corner. Therefore, we tried arithmetic, geometric and logarithmic mean and found, that for the average length $\bar{s}$ being the geometric mean of the maximum and minimum distance between the inner and outer surface the resulting first estimating expression for the shape factor

$$S_{1,sc} \cong \frac{2\pi}{\ln(x)+\ln\left(\frac{4}{\pi}\right)} \cdot \frac{\frac{4}{\pi}x-1}{\sqrt{(x-1)(\sqrt{2}x-1)}} \tag{15}$$

fits our numerical calculations best.

Replacing some numerical values by coefficients C_i gives

$$S_{1,sc} = \frac{2\pi}{\ln(x)+C_1} \cdot \frac{C_2x-1}{\sqrt{(x-1)(C_3x-1)}}. \tag{16}$$

For x tending to infinity equation (16) must be equal to the equation for a point source within the centre of a square, of which the analytical solution is [2]

$$S_{1,sp} = \frac{2\pi}{\ln(x)+C_1} \tag{17}$$

with C_1 being an asymptotic value of an infinite series found by the method of conformal mapping [6]. Considering the first five terms of the series solution $C_1 = 0.07577$ is found [6]. In this case the second factor of equation (16) has to tend to unity, which is fulfilled if

$$C_3 = C_2^2. \tag{18}$$

C_2 was fitted to the numerical results obtained by FEM with the method of least squares minimisation. This leads to the following equation for the shape factor as a function of x:

$$S_{1,sc} = \frac{2\pi}{\ln(x)+C_1} \cdot \frac{C_2x-1}{\sqrt{(x-1)(C_2^2x-1)}} \quad \text{with} \quad \begin{matrix} C_1 = 0.07577 \\ C_2 = 1.08140 \end{matrix}. \tag{19}$$

Equation (19) describes our numerical results with a maximum relative error better than 0.032 and for $x > 1.5$ better than 0.005. The maximum number of elements in the mesh was 6400 ($n = 80$). The relative error of the numerical calculations can be estimated to about the same as for the case of two concentric circles (see section 2.1 and Fig. 4), whereas an analytical solution is available for comparison.

In Fig. 6 the shape resistance as a function of x is presented. As a comparison measured values from Smith *et al.* [4] are shown. Their empirical correlation, also shown in Fig. 6, does not fulfil the condition that the shape factor has to become infinity ($R_1 \to 0$) for x tending to one.

3.3. *Square within the centre of a circle*

The average length $\bar{s}$ is obtained as the logarithmic mean of the maximum and minimum distance between the inner and outer surface, because this fits the numerical calculated values better than the geometric or the arithmetic mean. The ratio x for the outer diameter to the inner diameter cannot be less than $\sqrt{2}$. This results in the following approximate expression for the shape factor:

$$S_{1,cs} \cong \frac{2\pi x-8}{\sqrt{2}-1} \cdot \frac{\ln\left(\frac{x-1}{x-\sqrt{2}}\right)}{\ln\left(\frac{\pi}{4}x\right)}. \tag{20}$$

Rewriting equation (20) and fitting two constants C_i by minimisation of the maximum relative error being less than 0.007 to the values obtained by the Method of Finite Elements yields and the asymptotic value from equation (10) yields

$$S_{1,cs} = \frac{2\pi}{\ln\left(\frac{\pi}{4}x\right)} \cdot \ln\left(\frac{x-1}{x-\sqrt{2}}\right) \cdot \frac{x-C_1}{C_2} \quad \text{with} \quad \begin{matrix} C_1 = 1.3180 \\ C_2 = 0.4213 \end{matrix}. \tag{21}$$

Equation (21) describes our numerical results with a maximum relative error better than 0.0045. The maximum number of elements in the mesh was 6400.

In Fig. 7 the shape resistance as a function of x is presented. Equation (15) does fulfil the condition that the shape factor has to become infinity ($R_1 \to 0$) for x tending to $\sqrt{2}$. Since it is not quite clear, if in this case an infinitesimal square is equal to an infinitesimal circle, the limiting case for x tending to infinity might not be equal to the problem of concentric circles and is therefore not considered as an asymptote like the point-source is in the problem before.

4. CONCLUSION

Numerical methods have been used to obtain the shape factor as a function of the ratio x of the outer diameter to the inner diameter for three differently shaped cross-sections. For a given value of x different values for the shape factor are obtained by varying grid density n. Exponential functions of n are fitted to the different values to obtain an asymptotic value ($n \to \infty$) for the shape factor. The shape factor that is defined as the ratio of an average surface area to an average distance can be obtained by simple geometric

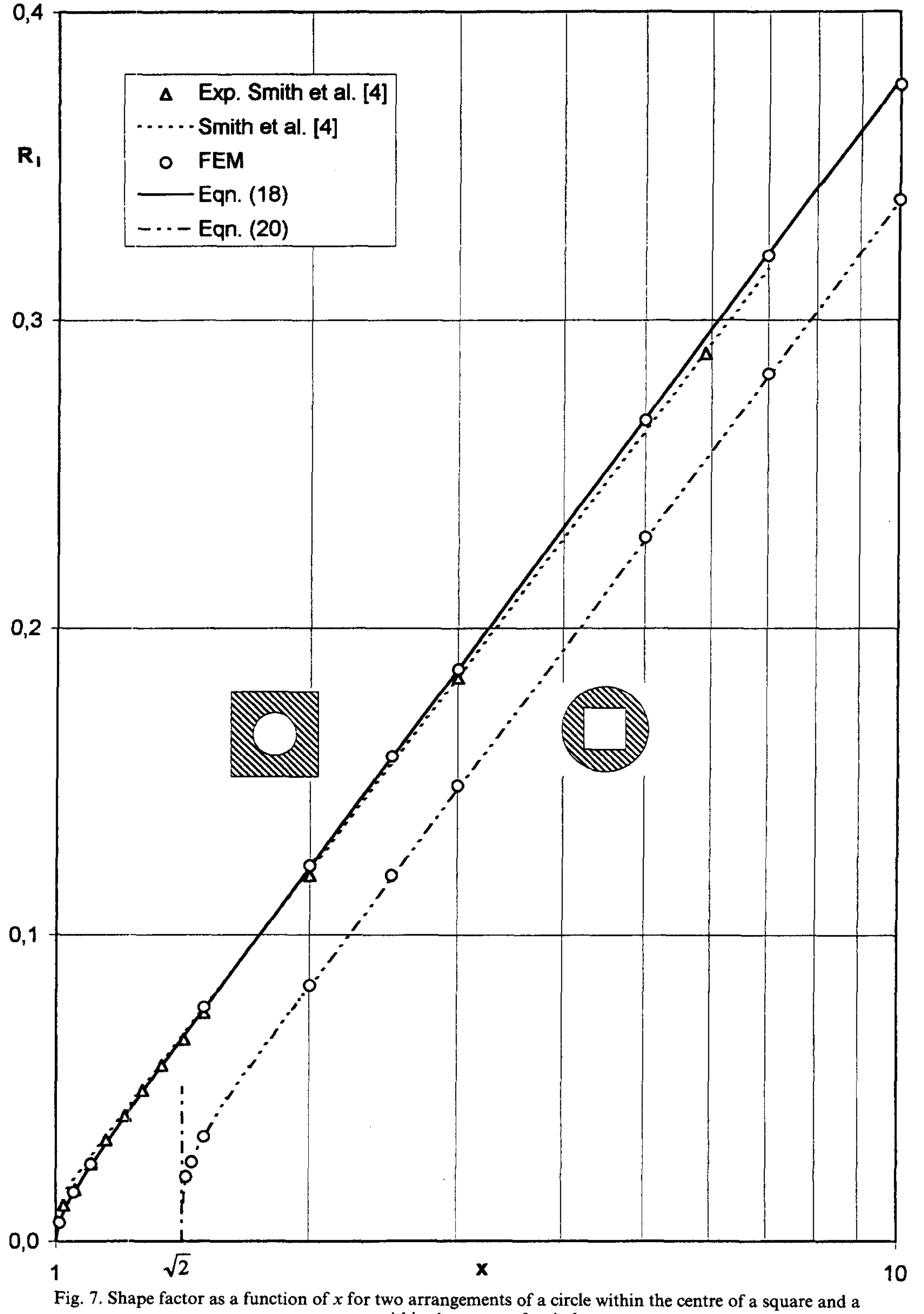

Fig. 7. Shape factor as a function of x for two arrangements of a circle within the centre of a square and a square within the centre of a circle.

assumptions leading to approximate analytical expressions. These expressions are slightly modified and their coefficients are fitted against numerical results for the investigated shapes. The equations obtained are in good agreement with experimental and numerical data from literature.

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